The Implications of Linear Independence on Free Abelian Groups

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Abstract
The existence of a basis in free abelian groups could not be separated from the concept of linear independence. In vector spaces, linear independence has several implications that play an important role in various basis discussions. The discussion regarding the free abelian groups focuses more on its basis and implications. In this article, to get deeper implications regarding the basis of free abelian groups, namely the implications of their linear independence first, such as if a subset X of F is linearly independent then every element of the subgroup generated by X can be written uniquely as a linear combination of some elements of X, and other implications. The methodology used in this article is literature study and focus group discussion to obtain information from algebra experts**.** The result of the research are there is an implication of linear independence in free abelian groups that applies as well as linear independence in vector space, but there are also certain implications that cannot be applied to free abelian groups. This will be interesting as further research regarding the properties of the basis in free abelian groups.

Keywords: Basis, Free Abelian Group, Implication, Linear Independence

1. Introduction

Real vector space is a concept in a branch of mathematics called linear algebra. Vector can be added to each other and multiplied by a scalar. As in Lax (2017), real vector space is an abelian group for addition operation and for scalar satisfy some conditions. Let V is a set of objects with two operations, let's call them addition and multiplication by scalars (real numbers). Scalar multiplication: for every scalar s and vector a ∈V is defined multiplication sa. Vector addition: for every vector a, b∈ V is defined a + b addition. So that, ∀a, b, c ∈ V and scalars s, t, and 1.

For all elements of a, b ∈ V then a + b ∈ V, the addition operation results in an element also in V. For any elements a, b, c ∈ V, addition operation is associative, meaning (a+b) + c = a + (b+c). There exist identity element e ∈ V such that for any element a, e + a =a + e = a. Every element a∈ V has inverse element -a ∈ V then a + (-a) = -a + a = e. The addition operation is commutative, meaning for all a, b ∈ V then a + b = b + a. For any scalars s, t∈ R and if elements a, b ∈ V, the operation satisfies the following properties: sa∈ V, s (ta) = (st) a, (s+t)a =sa + ta, s(a+b) =sa + sb, and 1a = a.

In other hand, as in Aliabadi (2022), abelian group for integer scalar satisfies same conditions as on real vector spaces. A commutative group, also called an abelian group, fulfils the following conditions 4 group requirements and the commutative property.

As in Dummit (2020), for all elements of a,b∈G then a + b ∈ G, the addition operation results in an element also in G. For any elements a, b, c ∈ G, addition operation is associative, meaning (a+b) + c = a + (b+c). There exist identity element e ∈ G such that for any element a, e + a = a + e = a. Every element a ∈ G has inverse element -a ∈ G then a + (-a) = -a + a = e. The addition operation is commutative, meaning for all a, b ∈ G then a + b = b + a. For any scalars s, t ∈ Z and if elements a, b ∈ G, the operation satisfies the following properties: sa ∈ G, s(ta) = (st)a, (s+t)a = sa + ta, s(a+b) = sa + sb, and 1a = a.

Commutative groups are like vector spaces but are not vector spaces. Although commutative groups and vector spaces have similarities, they are distinct mathematical structures with unique properties. Both involve a set of elements, and one or more operations defined on those elements. However, the specific operations and their properties set them apart.

In the discussion of vector spaces, there is a very important concept, namely the basis. Through basis, many things in the discussion of vector spaces can be found more easily, such as building all vectors in the vector space, getting a linear transformation representation matrix etc.

As in Axler (2015), in real vector space, there is subset that can be a basis, it is subset which can generate all vectors and subset which all the elements are linearly independent. As well as in the vector spaces, abelian group has basis too but not all abelian group has a basis. There is an abelian group which has a basis called free abelian group.

Table 1. Basis in Real Vector Space and Abelian Group

Real Vector Space	Free Abelian Group
The linear independence of a basis of a vector space has the following implications: 1) X is linearly independent if and only if every vector in V can be uniquely expressed in the form $k_1x_1 + k_2x_2 + \dots + k_nx_n$ 2) If V has dimension n, then every linearly independent subset of V consisting of n elements forms a basis. 3) If V is a vector space, then every linearly independent subset of V can be extended to form a basis. 4) If V is a vector space, then every subset that spans V contains a basis of V."	It will be checked whether the basis independence has criteria similar to the basis in vector space

2. Research Method

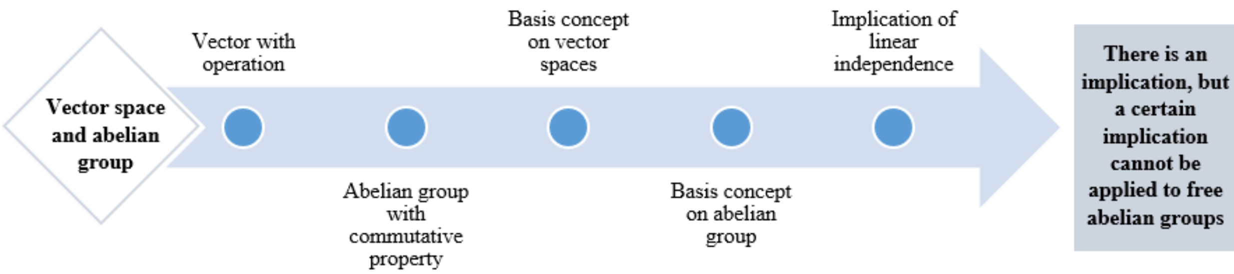
This research uses a type of literature study research (literature review) with the selected review model is a narrative review. The study conducted in the narrative review model is to compare data from several international journals that have been analysed and summarised based on the author's experience, existing theories and models.

The research method used is a qualitative research method with data sources used in the form of data obtained from several international journals, articles and previous studies that have been analysed by the author related to the problems to be studied in this study.

To obtain information that is not specified in the literature study related to the implications of linear independence on free abelian groups, focus group discussion (FGD) activities are carried out with mathematics experts in FGD I and FGD II.

Figure 1. Steps of The Research

Figure 2. Road Map of The Implementation of The Focus Group Discussion



Before we discuss about results and discussions, here are some definitions and theorems that will be used.

When discussing groups, we often use multiplication as the main notation. In this article, since we are discussing about abelian groups, each structure will be adjusted using the addition operation. For example, operation ab is replaced by a+b, invers a^(-1) is replaced by -a, identity e is replaced by 0, the n-th power a^n is replaced by na, and so on.

Theorem 2.1

For any abelian group G

a) $(m + n)a = ma + na$, for every $a \in G; m, n \in \mathbb{Z}$
b) $m(a + b) = ma + mb$, for every $a, b \in G; m \in \mathbb{Z}$.

Theorem 2.1 confirmed the properties of any abelian group that similar with the properties of real vector spaces about scalar multiplications.

Theorem 2.2

The subgroup ⟨X⟩ of an abelian group G consists of all elements with the form

$$k_1x_1 + k_2x_2 + \dots + k_nx_n,$$

for k_i ∈ Z and x_1, x_2, …, x_n are distinct elements of X. The form k_1 x_1 + k_2 x_2 + … + k_n x_n is called linear combination. In particular, the cylic group ⟨x⟩ is consists of all elements with the form nx for n ∈ Z.

Note that from Theorem 2.2, we have the definition of linear combination as exactly as in real vector space.

Definition 2.3

A subset X of an abelian group G is said to be linearly independent if

$$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$$

always implies $k_i = 0 \in \mathbb{Z}$ for all i, and x_1, x_2, ..., x_n are distinct elements of X.

Same as linear combination, from Definition 2.3, we have the definition of linearly independent as exactly as in real vector space.

Definition 2.4

A subset X of an abelian group G is said to be a basis if X is generating G and linearly independent.

Finally, from the Definition 2.4, the definition of basis as in real vector space has the similar condition to the definition of a basis for a free abelian group.

Definition 2.5

An abelian group G is called *free abelian group* if G has a basis.

From the definition, as mentioned above in Introduction, not every abelian group has a basis. So from Definition 2.4, we use free abelian group as the main topic so that we can compare their properties with real vector spaces.

Example 2.6

An abelian group $\mathbb{Z}$ is free abelian group with basis $X = \{1\}$.
Note that $n.1 = 0$ always implies $n = 0$, so X is linearly independent. Let z be any element of $\mathbb{Z}$. Note that $z = z.1 \in \langle 1 \rangle$, then X generates $\mathbb{Z}$. This is proving that X is basis of $\mathbb{Z}$.

Example 2.7

A group of modulo $n \neq 0$, $\mathbb{Z}_n$ is not free abelian group.
Let x be any element of $\mathbb{Z}_n$ and $kx = 0 \in \mathbb{Z}_n$. Note that for $k = n \neq 0$, $kx = nx = 0$. Then there is no element of $\mathbb{Z}_n$ that linearly independent. This is proving that $\mathbb{Z}_n$ has no basis which means $\mathbb{Z}_n$ is not free abelian group.

Example 2.8

A group of direct sum $\mathbb{Z} \oplus \mathbb{Z}$ is free abelian group with basis $X = \{(1,0), (0,1)\}$.
Note that $k(1,0) + l(0,1) = (0,0)$ always implies $k = l = 0$, so X is linearly independent. Let (a, b) be any element of $\mathbb{Z} \oplus \mathbb{Z}$. Note that $(a, b) = a(1,0) + b(0,1) \in \langle X \rangle$, then X generates $\mathbb{Z} \oplus \mathbb{Z}$. This is proving that X is basis of $\mathbb{Z} \oplus \mathbb{Z}$.

3. Results and Discussions

Based on the results of literature studies and discussions with algebra experts, the linear independence of the basis of a free abelian group does not have the same implications as the linear independence of real vector spaces. In this section, we explain each implication mentioned in the Introduction. The following theorem is the only implication of each implication that is satisfied.d.

Theorem 3.1

Any subset X of a free abelian group G is linearly independent if and only if every nonzero element of the subgroup $\langle X \rangle$ can be uniquely expressed in the form $k_1x_1 + k_2x_2 + \dots + k_nx_n$, for $k_i \in \mathbb{Z}$ and $x_1, x_2, \dots, x_n$ are distinct elements of X.

Proof:

Let X be any subset of a free abelian group G and linearly independent.

Let $u \in \langle X \rangle$ arbitrary, then $u = k_1x_1 + k_2x_2 + \dots + k_nx_n$, for $k_i \in \mathbb{Z}$ and $x_1, x_2, \dots, x_n$ are distinct elements of X.

Let $u = l_1x_1 + l_2x_2 + \dots + l_mx_m$ be another form of u. Without loss of generality, then

$$k_1x_1 + k_2x_2 + \dots + k_nx_n = l_1x_1 + l_2x_2 + \dots + l_mx_m + 0x_{m+1} + \dots + 0x_n$$

$$(k_1 - l_1)x_1 + (k_2 - l_2)x_2 + \dots + (k_m - l_m)x_m + k_{m+1}x_{m+1} + \dots + k_nx_n = 0_G.$$

Since X is linearly independent, then for $i = 1, \dots, m$, $k_i - l_i = 0$, and
for $i = m + 1, \dots, n$, $k_i = 0$.

So, $k_i = l_i$, proving that the expression is unique.

Conversely, let every nonzero element of the subgroup $\langle X \rangle$ can be uniquely expressed in the form $k_1x_1 + k_2x_2 + \dots + k_nx_n$, for $k_i \in \mathbb{Z}$ and $x_1, x_2, \dots, x_n$ are distinct elements of X.

Let $k_1x_1 + k_2x_2 + \dots + k_nx_n = 0_G$.

Since $0_G = 0x_1 + 0x_2 + \dots + 0x_n$, then $k_i = 0$, proving that X is linearly independent.

First Failed Implication

Some other implications mentioned in the introduction unfortunately cannot be satisfied by the linear independence of the basis of free abelian group. The first implication that fails is that "if G is a free abelian group of rank n, then every linearly independent subset of n elements of G is a basis". For example, consider the following free abelian additive group Z of rank 1.

Consider $X = \{2\} \subset \mathbb{Z}$. Note that $n.2 = 0$ if and only if $n = 0$, which means that X is linearly independent. However, X is not a basis of $\mathbb{Z}$ since $\exists 3 \in \mathbb{Z}$, but $3 = n.2$ has no integer solutions, or in other words $3 \neq n.2$ for every $n \in \mathbb{Z}$.

Second Failed Implication

A further implication that is not satisfied is that "if G is a free abelian group, then every linearly independent subset of G can be extended into a basis of G". For example, consider the following free abel group $\mathbb{Z} \oplus \mathbb{Z}$. Consider the subset $\{(2,0)\}$ of $\mathbb{Z} \oplus \mathbb{Z}$.

Note that $n.(2,0) = (0,0) \iff (2n, 0) = (0,0) \iff n = 0$, which means $\{(2,0)\}$ is linearly independent.

Since rank of $\mathbb{Z} \oplus \mathbb{Z}$ is 2, suppose that the subset $\{(2,0)\}$ can be extended to $\{(2,0), (a, b)\}$ as a basis of $\mathbb{Z} \oplus \mathbb{Z}$. Then for every $(x, y) \in \mathbb{Z} \oplus \mathbb{Z}$ can be written as a linear combination

$$(x, y) = k_1(2,0) + k_2(a, b)$$

with k_1 and k_2 be integers.

If we dissect the linear combination then $x = 2k_1 + ak_2$ and $y = bk_2$, which produce $k_2 = y/b$ and $k_1 = (x - \frac{ay}{b})/2$.

Since y is arbitrary and $k_2 \in \mathbb{Z}$, then it must be $b = 1$. Consequently, $k_1 = (x - ay)/2$. This requires that either x and ay are both even or x and ay are both odd. However, since x and y are arbitrary, they can be both even, both odd, or any other condition.

Consider the case when x is odd and y is even, then the condition x is odd and ay is even is created, in other words, $x - ay = (2p + 1) - a(2q) = 2(p - aq) + 1$ which is an odd number, as a result k_1 is not an integer, a contradiction.

Thus, $\{(2,0)\}$ cannot be extended into a basis of $\mathbb{Z} \oplus \mathbb{Z}$.

Third Failed Implication

The last incorrect implication is that "if G is a free abelian group, then every generating set of G contains a basis of F". For example, consider the free abelian group of integer polynomials of degree 1

$$P = \{a_0 + a_1X \mid a_i \in \mathbb{Z}\}$$

which has rank of 2.

Consider $A = \{2, 3, X\} \subset P$. Note that for every element $a_0 + a_1X \in P$

$$a_0 + a_1X = -a_0(2) + a_0(3) + a_1(X)$$

then A generates P. But note that

1. $A = \{2, 3, X\} \subseteq A$ is not linearly independent.
   Note that for $k_1 = -3$, $k_2 = 2$, and $k_3 = 0$, then
   $$k_1 2 + k_2 3 + k_3 X = -3(2) + 2(3) + 0X = -6 + 6 + 0 = 0$$
   Thus, $k_1 2 + k_2 3 + k_3 X = 0$ has solution which is not only $k_i = 0$.
   This shows that A is not linearly independent.
2. $\{2, 3\} \subset A$ is not linearly independent.
   Note that for $k_1 = -3$, and $k_2 = 2$ then
   $$k_1 2 + k_2 3 = -3(2) + 2(3) = -6 + 6 = 0$$
   Thus, $k_1 2 + k_2 3 = 0$ has solution which is not only $k_i = 0$.
   This shows that $\{2,3\}$ is not linearly independent.
3. $\{2, X\} \subset A$ cannot generates P.
   Note that for $1 + X \in P$
   $$1 + X \neq k_1(2) + k_2X$$
   for every $k_1, k_2 \in \mathbb{Z}$, This is because $k_1(2)$ always form an even number, so it is impossible to produce the number 1 on $1 + X$.
   This shows that $\{2, X\}$ does not generate P.
4. $\{3, X\} \subset A$ cannot generates P.
   Note that for $1 + X \in P$
   $$1 + X \neq k_1(3) + k_2X$$
   for every $k_1, k_2 \in \mathbb{Z}$, This is because $k_1(3)$ always form a multiple of 3, so it is impossible to produce the number 1 on $1 + X$.
   This shows that $\{3, X\}$ does not generate P.
5. Neither $\{2\}$, $\{3\}$, nor $\{X\}$ can generate P.

Based on the properties fulfilled by each subset of A above, it can be seen that none of them can be a basis. Thus, A does not contain a basis of P even though it can generates P.

4. Conclusions

There is an implication of linear independence in free abelian groups that is applied as well as linear independence in vector space. However, a certain implication cannot be applied to free abelian groups. This will be interesting for further research regarding the properties of the basis in free abelian groups whether they will be the same as the properties of the basis in real vector space or not.

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